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Generalized Cesàro operators in weighted Banach spaces of analytic functions with sup-norms

Angela A. Albanese[1] · **José Bonet**[2] · **Werner J. Ricker**[3]

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Abstract
An investigation is made of the generalized Cesàro operators C_t, for $t \in [0, 1]$, when they act on the space $H(\mathbb{D})$ of holomorphic functions on the open unit disc $\mathbb{D}$, on the Banach space H^∞ of bounded analytic functions and on the weighted Banach spaces H_v^∞ and H_v^0 with their sup-norms. Of particular interest are the continuity, compactness, spectrum and point spectrum of C_t as well as their linear dynamics and mean ergodicity.

Keywords Generalized Cesáro operator · Weighted Banach spaces of analytic functions · Compact operator · Spectrum · Supercyclic · Mean ergodic · Power bounded

Mathematics Subject Classification Primary 46E15, 47B38; Secondary 46E10, 47A10, 47A16, 47A35

1 Introduction and preliminaries

The (discrete) generalized Cesàro operators C_t, for $t \in [0, 1]$, were first investigated by Rhaly [25, 26]. The action of C_t from the sequence space $\omega := \mathbb{C}^{\mathbb{N}_0}$ into itself, with $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, is given by

$$C_t x := \left(\frac{t^n x_0 + t^{n-1} x_1 + \cdots + x_n}{n+1} \right)_{n \in \mathbb{N}_0}, \quad x = (x_n)_{n \in \mathbb{N}_0} \in \omega. \tag{1.1}$$

✉ Angela A. Albanese
angela.albanese@unisalento.it

José Bonet
jbonet@mat.upv.es

Werner J. Ricker
werner.ricker@ku.de

1 Dipartimento di Matematica e Fisica “E. De Giorgi”, Università del Salento, C.P.193, 73100 Lecce, Italy

2 Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politécnica de Valencia, Cubo F, Cuarta Planta, 46071 Valencia, Spain

3 Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, 85072 Eichstätt, Germany

For $t = 0$ and with $\varphi := (\frac{1}{n+1})_{n\in\mathbb{N}_0}$ note that C_0 is the diagonal operator

$$D_\varphi x := \left(\frac{x_n}{n+1}\right)_{n\in\mathbb{N}_0}, \quad x = (x_n)_{n\in\mathbb{N}_0} \in \omega, \tag{1.2}$$

and, for $t = 1$, that C_1 is the classical Cesàro averaging operator

$$C_1 x := \left(\frac{x_0 + x_1 + \cdots + x_n}{n+1}\right)_{n\in\mathbb{N}_0}, \quad x = (x_n)_{n\in\mathbb{N}_0} \in \omega. \tag{1.3}$$

The behaviour of C_t on various sequence spaces has been investigated by many authors. We refer the reader to [25–27], to the recent papers [28, 30, 31] and to the introduction of the papers [5, 13] and the references therein. The operator C_1 was thoroughly investigated on weighted Banach spaces in [2]; see also [12]. Certain variants of the Cesàro operator C_1 are considered in [9, 16].

Our aim is to investigate the operators C_t, for $t \in [0, 1]$, when they are suitably interpreted to act on the space $H(\mathbb{D})$ of holomorphic functions on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, on the Banach space H^∞ of bounded analytic functions and on the weighted Banach spaces H_v^∞ and H_v^0 with their sup-norms. The space $H(\mathbb{D})$ is equipped with the topology τ_c of uniform convergence on the compact subsets of $\mathbb{D}$. According to [21, §27.3(3)] the space $H(\mathbb{D})$ is a Fréchet–Montel space. A family of norms generating τ_c is given, for each $0 < r < 1$, by

$$q_r(f) := \sup_{|z|\le r} |f(z)|, \quad f \in H(\mathbb{D}). \tag{1.4}$$

A *weight* v is a continuous, non-increasing function $v\colon [0, 1) \to (0, \infty)$. We extend v to $\mathbb{D}$ by setting $v(z) := v(|z|)$, for $z \in \mathbb{D}$. Note that $v(z) \le v(0)$ for all $z \in \mathbb{D}$. Given a weight v on $[0, 1)$, we define the corresponding *weighted Banach spaces of analytic functions* on $\mathbb{D}$ by

$$H_v^\infty := \{f \in H(\mathbb{D}) : \|f\|_{\infty,v} := \sup_{z\in\mathbb{D}} |f(z)|v(z) < \infty\},$$

and

$$H_v^0 := \{f \in H(\mathbb{D}) : \lim_{|z|\to 1^-} |f(z)|v(z) = 0\},$$

both endowed with the norm $\|\cdot\|_{\infty,v}$. Since $\|f\|_{\infty,v} \le v(0)\|f\|_\infty$ whenever $f \in H^\infty$, it is clear that $H^\infty \subseteq H_v^\infty$ with a continuous inclusion. If $v(z) = 1$ for all $z \in \mathbb{D}$, then H_v^∞ coincides with the space H^∞ of all bounded analytic functions on $\mathbb{D}$ with the sup-norm $\|\cdot\|_\infty$ and H_v^0 reduces to $\{0\}$. Moreover, $H_v^\infty \subseteq H(\mathbb{D})$ continuously. Indeed, fix $0 < r < 1$. Then $\frac{1}{v(0)} \le \frac{1}{v(z)} \le \frac{1}{v(r)}$ for $|z| \le r$ and so (1.4) implies that

$$q_r(f) = \sup_{|z|\le r} \frac{v(z)|f(z)|}{v(z)} \le \frac{1}{v(r)} \sup_{|z|\le r} v(z)|f(z)| \le \frac{1}{v(r)}\|f\|_{\infty,v}, \quad f \in H_v^\infty.$$

We refer the reader to [10] for a recent survey of such types of weighted Banach spaces and operators between them.

Whenever necessary we will identify a function $f \in H(\mathbb{D})$ with its sequence of Taylor coefficients $\hat{f} := (\hat{f}(n))_{n\in\mathbb{N}_0}$ (i.e., $\hat{f}(n) := \frac{f^{(n)}(0)}{n!}$, for $n \in \mathbb{N}_0$), so that $f(z) = \sum_{n=0}^\infty \hat{f}(n)z^n$, for $z \in \mathbb{D}$. The linear map $\Phi\colon H(\mathbb{D}) \to \omega$ is defined by

$$\Phi\left(f = \sum_{n=0}^\infty \hat{f}(n)z^n\right) := \hat{f}, \quad f \in H(\mathbb{D}).$$

It is injective (clearly) and continuous. Indeed, for each $m \in \mathbb{N}_0$,

$$r_m(x) := \max_{0\le j\le m} |x_j|, \quad x = (x_j)_{j\in\mathbb{N}_0} \in \omega,$$

is a continuous seminorm in ω. Fix $0 < r < 1$, in which case

$$r_m(\Phi(f)) = \max_{0\le j\le m} |\hat{f}(j)| = \max_{0\le j\le m} \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{j+1}}\, dz \right| \le \max_{0\le j\le m} \sup_{|z|=r} \frac{|f(z)|}{|z|^j}$$
$$= \max_{0\le j\le m} \frac{1}{r^j} q_r(f) = \frac{1}{r^m} q_r(f),$$

for each $f \in H(\mathbb{D})$ because $\frac{1}{r^j} \le \frac{1}{r^m}$ for all $0 \le j \le m$. Of course, the increasing sequence of seminorms $\{r_m \ m \in \mathbb{N}_0\}$ generates the topology of ω.

We first provide an integral representation of the generalized Cesàro operators C_t defined on $H(\mathbb{D})$, for $t \in [0, 1)$. So, fix $t \in [0, 1)$ and define $C_t\colon H(\mathbb{D}) \to H(\mathbb{D})$ by $C_t f(0) := f(0)$ and

$$C_t f(z) := \frac{1}{z} \int_0^z \frac{f(\xi)}{1-t\xi}\, d\xi, \ z \in \mathbb{D} \setminus \{0\}, \tag{1.5}$$

for every $f \in H(\mathbb{D})$. It turns out that C_t is continuous on $H(\mathbb{D})$; see Proposition 2.1. Moreover, the discrete Cesàro operator $C_t\colon \omega \to \omega$, when restricted to the subspace $\Phi(H(\mathbb{D})) \subseteq \omega$ is transferred to $H(\mathbb{D})$ as follows. For a fixed $f \in H(\mathbb{D})$ we have $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$, for $\xi \in \mathbb{D}$, with $\hat{f} = (a_n)_{n\in\mathbb{N}_0}$ its sequence of Taylor coefficients. Since $\frac{1}{1-t\xi} = \sum_{n=0}^{\infty} t^n \xi^n$, for $\xi \in \mathbb{D}$, we can form the Cauchy product of the two series, thereby obtaining

$$\frac{f(\xi)}{1-t\xi} = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} t^{n-k} a_k) \xi^n, \quad \xi \in \mathbb{D}.$$

Then (1.5) yields

$$zC_t f(z) = \int_0^z \sum_{n=0}^{\infty} (\sum_{k=0}^{n} t^{n-k} a_k) \xi^n\, d\xi = \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \cdots + a_n}{n+1} \right) z^{n+1}, \ z \in \mathbb{D}.$$

The interchange of the infinite sum and the integral is permissible by uniform convergence of the series. This shows that $C_t f \in H(\mathbb{D})$ also has the series representation

$$C_t f(z) = \sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1} a_1 + \cdots + a_n}{n+1} \right) z^n$$
$$= \sum_{n=0}^{\infty} \left(\frac{t^n \hat{f}(0) + t^{n-1} \hat{f}(1) + \cdots + \hat{f}(n)}{n+1} \right) z^n = \sum_{n=0}^{\infty} (C_t^{\omega}(\hat{f}))_n z^n, \tag{1.6}$$

where the coefficients of the series are precisely as in (1.1). For the sake of clarity we will denote the discrete generalized Cesàro operator $C_t\colon \omega \to \omega$ by C_t^{ω} and reserve the notation C_t for the operator (1.5) acting in $H(\mathbb{D})$. Note that $C_0^{\omega} = D_{\varphi}$ (see (1.2)). Moreover, C_0 is given by $C_0 f(z) = \frac{1}{z} \int_0^z f(\xi)\, d\xi$ for $z \neq 0$ and $C_0 f(0) = f(0)$, which is the classical Hardy operator in $H(\mathbb{D})$.

The main results for C_t when acting in the Fréchet space $H(\mathbb{D})$ occur in Proposition 2.1 (continuity), Proposition 3.3 (non-compactness), Proposition 3.7 (spectra) and Proposition 3.8 (linear dynamics and mean ergodicity). For the analogous information concerning C_t when acting in the weighted Banach spaces H_v^{∞} and H_v^0 see Proposition 2.4 and Corollary

2.5 (continuity), Proposition 2.7 (compactness), Proposition 2.8 (spectra) and Proposition 3.2 (linear dynamics and mean ergodicity).

We end this section by recalling a few definitions and some notation concerning locally convex spaces and operators between them. For further details about functional analysis and operator theory relevant to this paper see, for example, [15, 18, 20–22, 29].

Given locally convex Haudorff spaces X, Y (briefly, lcHs) we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X into Y. If $X = Y$, then we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. Equipped with the topology of pointwise convergence on X (i.e., the strong operator topology) the lcHs $\mathcal{L}(X)$ is denoted by $\mathcal{L}_s(X)$. Equipped with the topology τ_b of uniform convergence on the bounded subsets of X the lcHs $\mathcal{L}(X)$ is denoted by $\mathcal{L}_b(X)$.

Let X be a lcHs space. The identity operator on X is denoted by I. The *transpose operator* of $T \in \mathcal{L}(X)$ is denoted by T'; it acts from the topological dual space $X' := \mathcal{L}(X, \mathbb{C})$ of X into itself. Denote by X'_σ (resp., by X'_β) the topological dual X' equipped with the weak* topology $\sigma(X', X)$ (resp., with the strong topology $\beta(X', X)$); see [21, §21.2] for the definition. It is known that $T' \in \mathcal{L}(X'_\sigma)$ and $T' \in \mathcal{L}(X'_\beta)$, [22, p. 134]. The bi-transpose operator $(T')'$ of T is simply denoted by T'' and belongs to $\mathcal{L}((X'_\beta)'_\beta)$.

A linear map $T: X \to Y$, with X, Y lcHs', is called *compact* if there exists a neighbourhood $\mathcal{U}$ of 0 in X such that $T(\mathcal{U})$ is a relatively compact set in Y. It is routine to show that necessarily $T \in \mathcal{L}(X, Y)$. We recall the following well known result; see [20, Proposition 17.1.1], [22, §42.1(1)].

Lemma 1.1 *Let X be a lcHs. The compact operators are a 2-sided ideal in $\mathcal{L}(X)$.*

Given a lcHs X and $T \in \mathcal{L}(X)$, the resolvent set $\rho(T; X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T; X) := \mathbb{C}\backslash\rho(T; X)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T; X)$ of T consists of all $\lambda \in \mathbb{C}$ (also called an eigenvalue of T) such that $(\lambda I - T)$ is not injective. Some authors (eg. [29]) prefer the subset $\rho^*(T; X)$ of $\rho(T; X)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that the open disc $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T; X)$ and $\{R(\mu, T) : \mu \in B(\lambda, \delta)\}$ is an equicontinuous subset of $\mathcal{L}(X)$. Define $\sigma^*(T; X) := \mathbb{C}\backslash\rho^*(T; X)$, which is a closed set with $\sigma(T; X) \subseteq \sigma^*(T; X)$. For the spectral theory of compact operators in lcHs' we refer to [15, 18], for linear dynamics to [6], [17] and for mean ergodic operators to [23], for example.

2 Continuity, compactness and spectrum of C_t

In this section we establish, for $t \in [0, 1)$, the continuity of $C_t: H(\mathbb{D}) \to H(\mathbb{D})$ as well as the continuity of C_t from H^∞ (resp., H_v^∞) into H^∞ (resp., H_v^∞). The same is true for $C_t: H_v^0 \to H_v^0$ whenever $\lim_{r\to 1^-} v(r) = 0$. It is also shown that the bi-transpose C_t'' of $C_t \in \mathcal{L}(H_v^0)$ is the generalized Cesàro operator $C_t \in \mathcal{L}(H_v^\infty)$, provided that $\lim_{r\to 1^-} v(r) = 0$. For such weights v it also turns out that both $C_t \in \mathcal{L}(H_v^0)$ and $C_t \in \mathcal{L}(H_v^\infty)$ are compact operators (cf. Proposition 2.7); their spectrum is identified in Proposition 2.8. Of particular interest are the standard weights $v_\gamma(z) := (1 - |z|)^\gamma$, for $\gamma > 0$ and $z \in \mathbb{D}$.

Proposition 2.1 *For every $t \in [0, 1)$ the operator $C_t: H(\mathbb{D}) \to H(\mathbb{D})$ is continuous. Moreover, the set $\{C_t : t \in [0, 1)\}$ is equicontinuous in $\mathcal{L}(H(\mathbb{D}))$.*

Proof Fix $f \in H(\mathbb{D})$. Taking into account that $C_t f(0) = f(0)$, for all $t \in [0, 1)$ and, for each $r \in (0, 1)$, that $\sup_{|z|\le r} |C_t f(z)| = \sup_{|z|=r} |C_t f(z)|$, the formula (1.5) implies, for each $z \in \mathbb{D}\setminus\{0\}$, that

$$|C_t f(z)| = \frac{1}{|z|}\left|\int_0^z \frac{f(\xi)}{1-t\xi}d\xi\right| \leq \frac{1}{|z|}|z| \max_{\xi\in[0,z]} \frac{|f(\xi)|}{|1-t\xi|}$$
$$\leq \frac{1}{1-|z|}\max_{|\xi|\leq|z|}|f(\xi)| = \frac{1}{1-|z|}\max_{|\xi|=|z|}|f(\xi)|,$$

because $|1-t\xi| \geq 1-t|\xi| \geq 1-|\xi| \geq 1-|z|$, for all $|\xi| \leq |z|$. It follows from the previous inequality, for each $r \in (0,1)$, that

$$q_r(C_t f) = \sup_{|z|\leq r}|C_t f(z)| \leq \frac{1}{1-r}\sup_{|\xi|\leq r}|f(\xi)| = \frac{1}{1-r}q_r(f);.$$

see (1.4). This implies the result. □

The following example will prove to be useful in the sequel.

Example 2.2 Consider the constant function $f_1(z) := 1$, for every $z \in \mathbb{D}$, in which case $C_t f_1(0) = f_1(0) = 1$ for every $t \in [0,1]$. For $t = 0$, it was noted in Sect. 1 that C_0 is
the Hardy operator. In particular, $C_0 f_1(z) = 1$, for every $z \in \mathbb{D}$. For $t \in (0,1]$, note that $C_t f_1(0) = 1$ and

$$C_t f_1(z) = \frac{1}{z}\int_0^z \frac{d\xi}{1-t\xi} = -\frac{1}{tz}\log(1-tz), \quad z \in \mathbb{D}\setminus\{0\}.$$

For $t = 1$ this shows, in particular, that $C_1(H^\infty) \not\subset H^\infty$, which is well known. For an investigation of the operator C_1 acting in H^∞ we refer to [14].

Concerning $t \in (0,1)$, recall the Taylor series expansion

$$-\log(1-z) = z\sum_{n=0}^{\infty}\frac{z^n}{n+1}, \quad z \in \mathbb{D},$$

from which it follows that

$$-\frac{\log(1-tz)}{tz} = \sum_{n=0}^{\infty}\frac{t^n}{n+1}z^n, \quad z \in \mathbb{D}\setminus\{0\},$$

with the series having radius of convergence $\frac{1}{t} > 1$. The claim is that $\|C_t f_1\|_\infty = \sup_{|z|<1}|C_t f_1(z)| = -\frac{\log(1-t)}{t}$. Indeed, $C_t f_1$ is clearly holomorhic in $B(0,\frac{1}{t}) := \{\xi \in \mathbb{C} : |\xi| < \frac{1}{t}\}$ hence, continuous in $B(0,\frac{1}{t})$, and satisfies $C_t f_1(1) = -\frac{\log(1-t)}{t}$ with $\lim_{r\to 1^-} C_t f_1(r) = C_t f_1(1)$. On the other hand, for every $z \in \mathbb{D}\setminus\{0\}$ and $t \in (0,1)$ we have that

$$|C_t f_1(z)| = \left|-\frac{\log(1-tz)}{tz}\right| \leq \sum_{n=0}^{\infty}\frac{t^n}{n+1}|z|^n \leq \sum_{n=0}^{\infty}\frac{t^n}{n+1} = -\frac{\log(1-t)}{t}.$$

This completes the proof of the claim. Observe that $\|C_t f_1\|_\infty > 1$. Indeed, define $\gamma(t) = -\log(1-t) - t$, for $t \in [0,1)$. Then $\gamma(0) = 0$, $\lim_{t\to 1^-}\gamma(t) = \infty$ and $\gamma'(t) = \frac{1}{1-t} - 1 = \frac{t}{1-t}$, for $t \in [0,1)$. Since $\gamma'(t) > 0$, for $t \in (0,1)$, it follows that γ is strictly increasing and so $\gamma(t) > 0$ for all $t \in (0,1)$. This implies that $\|C_t f_1\|_\infty = -\frac{\log(1-t)}{t} > 1$ for every

$t \in (0,1)$. On the other hand, for $t \in (0,1)$, the inequality $\sum_{n=0}^{\infty} t^n/(n+1) < \sum_{n=0}^{\infty} t^n$ implies that $-\frac{\log(1-t)}{t} < \frac{1}{1-t}$. So, we have shown that $\|C_0 f_1\|_\infty = 1$ and

$$1 < \|C_t f_1\|_\infty < \frac{1}{1-t}, \quad t \in (0,1).$$

We now turn to the action of C_t in various Banach spaces. For $t = 1$ it was noted above that C_1 fails to act in H^∞.

Proposition 2.3 *For $t \in [0,1)$ the operator $C_t\colon H^\infty \to H^\infty$ is continuous. Moreover, $\|C_0\|_{H^\infty \to H^\infty} = 1$ and*

$$\|C_t\|_{H^\infty \to H^\infty} = -\frac{\log(1-t)}{t}, \quad t \in (0,1).$$

Proof Let $f \in H^\infty$ be fixed. Then

$$|C_0 f(z)| = \left| \frac{1}{z} \int_0^z f(\xi) d\xi \right| \le \max_{|\xi| \le |z|} |f(\xi)| \le \|f\|_\infty.$$

This implies that $\|C_0\|_{H^\infty \to H^\infty} \le 1$. On the other hand, $C_0 f_1 = f_1$ and so we can conclude that $\|C_0\|_{H^\infty \to H^\infty} = 1$.

Now let $t \in (0,1)$. Then, for the parametrization $\xi := sz$, for $s \in (0,1)$, it follows from $|1 - stz| \ge 1 - |stz| \ge 1 - st$ that

$$\begin{aligned} |C_t f(z)| &= \left| \frac{1}{z} \int_0^z \frac{f(\xi)}{1 - t\xi} d\xi \right| = \left| \int_0^1 \frac{f(sz)}{1 - stz} ds \right| \le \max_{|\xi| \le |z|} |f(\xi)| \int_0^1 \frac{ds}{1 - st|z|} \\ &\le \|f\|_\infty \int_0^1 \frac{ds}{1 - st} = -\frac{\log(1-t)}{t} \|f\|_\infty. \end{aligned}$$

So, $C_t \in \mathcal{L}(H^\infty)$ with $\|C_t\|_{H^\infty \to H^\infty} \le -\frac{\log(1-t)}{t}$. But, $\|C_t f_1\|_\infty = -\frac{\log(1-t)}{t}$. Accordingly, $\|C_t\|_{H^\infty \to H^\infty} = -\frac{\log(1-t)}{t}$. □

Proposition 2.4 *Let v be a weight function on $[0,1)$. For each $t \in [0,1)$ the operator $C_t\colon H_v^\infty \to H_v^\infty$ is continuous. Moreover, $\|C_0\|_{H_v^\infty \to H_v^\infty} = 1$ and*

$$1 \le \|C_t\|_{H_v^\infty \to H_v^\infty} \le -\frac{\log(1-t)}{t}, \quad t \in (0,1).$$

Proof Recall that $C_t f(0) := f(0)$ for each $f \in H(\mathbb{D})$ and $t \in [0,1]$. Fix $t \in (0,1)$. Given $f \in H_v^\infty$ and $z \in \mathbb{D}\setminus\{0\}$, observe that

$$\begin{aligned} v(z)|C_t f(z)| &= \frac{v(z)}{|z|} \left| \int_0^z \frac{f(\xi)}{1 - t\xi} d\xi \right| = v(z) \left| \int_0^1 \frac{f(sz)}{1 - stz} ds \right| \\ &\le v(z) \int_0^1 \frac{|f(sz)|}{|1 - stz|} ds \le \int_0^1 \frac{v(sz)|f(sz)|}{|1 - stz|} ds \\ &\le \|f\|_{\infty,v} \int_0^1 \frac{ds}{|1 - stz|} \le \|f\|_{\infty,v} \int_0^1 \frac{ds}{1 - st|z|} \\ &= -\frac{\log(1 - t|z|)}{t|z|} \|f\|_{\infty,v}, \end{aligned}$$

where we used that $v(sz) = v(s|z|) \geq v(|z|) = v(z)$, for $s \in (0, 1)$, as v is non-increasing on $(0, 1)$ and that $|1 - stz| \geq 1 - st|z|$, for $s \in (0, 1)$. According to the calculations in Example 2.2 we can conclude that

$$\|C_t f\|_{\infty,v} = \sup_{z\in\mathbb{D}} |C_t f(z)|v(z) \leq \|f\|_{\infty,v} \sup_{z\in\mathbb{D}} \left[-\frac{\log(1 - t|z|)}{t|z|}\right] = -\frac{\log(1 - t)}{t}\|f\|_{\infty,v}.$$

This implies that $C_t \in \mathcal{L}(H_v^\infty)$ and $\|C_t\|_{H_v^\infty \to H_v^\infty} \leq -\frac{\log(1-t)}{t}$.

For $t = 0$ observe that

$$|C_0 f(z)| \leq \int_0^1 |f(sz)|ds \leq \max_{|\xi|\leq|z|} |f(\xi)| = \frac{1}{v(z)} \max_{|\xi|=|z|} |f(\xi)|v(\xi) \leq \frac{1}{v(z)}\|f\|_{\infty,v},$$

as $v(\xi) = v(z)$ whenever $|\xi| = |z|$ with $\xi \in \mathbb{D}$. This shows that $\|C_0\|_{H_v^\infty \to H_v^\infty} \leq 1$. Since $C_0 f_1 = f_1$, it follows that actually $\|C_0\|_{H_v^\infty \to H_v^\infty} = 1$.

It remains to show that $\|C_t\|_{H_v^\infty \to H_v^\infty} \geq 1$ for $t \in (0, 1)$. To this end, fix $t \in (0, 1)$ and consider the function $g_0(z) := \frac{1}{1-tz} = \sum_{n=0}^\infty t^n z^n$, for $z \in \mathbb{D}$. Then $\|g_0\|_\infty = \frac{1}{1-t}$ and so $g_0 \in H^\infty \subseteq H_v^\infty$. Moreover, for every $z \in \mathbb{D}\setminus\{0\}$, it is the case that

$$C_t g_0(z) = \frac{1}{z}\int_0^z \frac{d\xi}{(1 - t\xi)^2} = \frac{1}{z}\left[\frac{1}{t(1 - t\xi)}\right]_0^z = \frac{1}{tz}\left[\frac{1}{1 - tz} - 1\right] = \frac{1}{1 - tz} = g_0(z).$$

It follows that $\|g_0\|_{\infty,v} = \|C_t g_0\|_{\infty,v} \leq \|C_t\|_{H_v^\infty \to H_v^\infty}\|g_0\|_{\infty,v}$ which implies that $\|C_t\|_{H_v^\infty \to H_v^\infty} \geq 1$. □

Corollary 2.5 *Let v be a weight function on $[0, 1)$ satisfying $\lim_{r\to 1^-} v(r) = 0$. For each $t \in [0, 1)$ the operator $C_t\colon H_v^0 \to H_v^0$ is continuous and satisfies $\|C_t\|_{H_v^0 \to H_v^0} = \|C_t\|_{H_v^\infty \to H_v^\infty}$.*

Proof By Proposition 2.4 and the fact that H_v^0 is a closed subspace of H_v^∞, to obtain the result it suffices to establish that $C_t(H_v^0) \subseteq H_v^0$. To this effect, observe that $H^\infty \subseteq H_v^0$ and that H^∞ is dense in H_v^0, as the space of polynomials is dense in H_v^0; see Section 1 of [11] and also [7]. Proposition 2.3 implies that $C_t(H^\infty) \subseteq H^\infty \subseteq H_v^0$. Since C_t acts continuously on H_v^∞, it follows that

$$C_t(H_v^0) = C_t(\overline{H^\infty}) \subseteq \overline{C_t(H^\infty)} \subseteq H_v^0.$$

Moreover, $\lim_{r\to 1^-} v(r) = 0$ implies that H_v^∞ is canonically isometric to the bidual of H_v^0, [8, Example 2.1], and that the bi-transpose $C_t''\colon H_v^\infty \to H_v^\infty$ of $C_t\colon H_v^0 \to H_v^0$ coincides with $C_t\colon H_v^\infty \to H_v^\infty$ (see Lemma 2.6 below), from which the identity $\|C_t\|_{H_v^0 \to H_v^0} = \|C_t\|_{H_v^\infty \to H_v^\infty}$ follows. □

Lemma 2.6 *Let v be a weight function on $[0, 1)$ satisfying $\lim_{r\to 1^-} v(r) = 0$. For each $t \in [0, 1)$, the bi-transpose $C_t''\colon H_v^\infty \to H_v^\infty$ of $C_t\colon H_v^0 \to H_v^0$ coincides with $C_t\colon H_v^\infty \to H_v^\infty$.*

Proof By Proposition 2.3 and Corollary 2.5, together with the fact that H_v^∞ is canonically isometric to the bidual of H_v^0, both of the operators C_t'' and C_t act continuously on H_v^∞.

To show that the bi-transpose $C_t''\colon H_v^\infty \to H_v^\infty$ of $C_t\colon H_v^0 \to H_v^0$ coincides with $C_t\colon H_v^\infty \to H_v^\infty$ we proceed via several steps.

First step Given $f \in H(\mathbb{D})$, its Taylor polynomials $p_k(z) = \sum_{j=0}^k \hat{f}(j)z^j$, $z \in \mathbb{D}$, for $k \in \mathbb{N}_0$, converge to f uniformly on compact subsets of $\mathbb{D}$. That is, $p_k \to f$ in $(H(\mathbb{D}), \tau_c)$ as $k \to \infty$. Accordingly, the averages of $(p_k)_{k\in\mathbb{N}_0}$, that is, the Cesàro means $f_n(z) := \frac{1}{n+1}\sum_{j=0}^n p_j(z)$, for $z \in \mathbb{D}$ and $n \in \mathbb{N}_0$, also converge to f in $(H(\mathbb{D}), \tau_c)$ as $n \to \infty$.

Second step Lemma 1.1 in [7] implies, for every $f \in H_v^\infty$ and $n \in \mathbb{N}_0$, that $\|f_n\|_{\infty,v} \leq \|f\|_{\infty,v}$, where f_n is the n-th Cesàro mean of f, as defined in the *First step*. Denote by U_v the closed unit ball of $(H_v^\infty, \|\cdot\|_{\infty,v})$. Then, for any given $f \in U_v$, its sequence of Cesàro means satisfies $(f_n)_{n\in\mathbb{N}_0} \subseteq U_v$ and $f_n \to f$ in $(H(\mathbb{D}), \tau_c)$ as $n \to \infty$.

Third step With the topology of uniform convergence on the compact subsets of U_v denoted by τ_c, let $X := \{F \in (H_v^\infty)' : F|_{U_v}$ is $\tau_c -$ continuous $\}$ be endowed with the norm $\|F\| := \sup\{|F(f)| : f \in U_v\}$. Then [8, Theorem 1.1(a)] ensures that $(X, \|\cdot\|)$ is a Banach space and that the evaluation map $\Psi\colon H_v^\infty \to X'$ defined by $(\Psi(f))(F) := \langle f, F\rangle$, for $F \in X$ and $f \in H_v^\infty$, is an isometric isomorphism onto X' (where X' is the dual Banach space of $(X, \|\cdot\|)$). Moreover, by [8, Theorem 1.1(b) and Example 2.1] the restriction map $R\colon X \to (H_v^0)'$ given by $F \mapsto F|_{H_v^0}$, is also a surjective isometric isomorphism. Therefore, the spaces H_v^∞ and $(H_v^0)''$ are isometrically isomorphic, that is, X and $(H_v^0)'$ are isometrically isomorphic and hence, also H_v^∞ and $(H_v^0)''$ are isometrically isomorphic.

It is easy to see, since the Banach space X above is the predual of H_v^∞, that the evaluation map $\delta_z \in X$, for every $z \in \mathbb{D}$, where $\delta_z\colon f \mapsto f(z)$, for $f \in H_v^\infty$, satisfies $|\langle f, \delta_z\rangle| \leq \|f\|_{\infty,v}/v(z)$. In particular, the linear span L of the set $\{\delta_z : z \in \mathbb{D}\}$ separates the points of $H_v^\infty = X'$ and hence, L is dense in X. Therefore, the pointwise convergence topology τ_p on H_v^∞ is Hausdorff and coarser than the w^*-topology $\sigma(H_v^\infty, X)$.

Fourth step The closed unit ball U_v of H_v^∞ is a τ_c-compact set by Montel's theorem, as it is τ_c-bounded and closed. On the other hand, U_v is also $\sigma(H_v^\infty, X)$-compact by the Alaoglu-Bourbaki theorem. Since $\tau_p|_{U_v}$ is coarser than $\tau_c|_{U_v}$ and Hausdorff, we can conclude that $\tau_p|_{U_v} = \tau_c|_{U_v}$. In the same way, it follows that $\tau_p|_{U_v} = \sigma(H_v^\infty, X)|_{U_v}$. Accordingly, $\tau_p|_{U_v} = \tau_c|_{U_v} = \sigma(H_v^\infty, X)|_{U_v}$.

We are now ready to prove that $(C_t)'' = C_t$. To show this, it suffices to establish that $(C_t)''f = C_t f$ for every $f \in U_v$.

So, fix $f \in U_v$. With $(f_n)_{n\in\mathbb{N}_0}$ as in the *First step* it follows from there that $f_n \to f$ in $(H(\mathbb{D}), \tau_c)$ as $n \to \infty$ and, by the *Second step*, that $(f_n)_{n\in\mathbb{N}_0} \subseteq U_v$. This implies that $C_t f_n \to C_t f$ in $(H(\mathbb{D}), \tau_c)$ as $n \to \infty$. Since $C_t \in \mathcal{L}(H_v^\infty)$ and $f \in U_v$, it is clear that $C_t f \in H_v^\infty$. On the other hand, by the *Fourth step* the sequence $(f_n)_{n\in\mathbb{N}_0}$ also converges to f in $(H_v^\infty, \sigma(H_v^\infty, X)) = (H_v^\infty, \sigma(H_v^\infty, (H_v^0)'))$. Since $(C_t)''\colon ((H_v^0)'', \sigma((H_v^0)'', (H_v^0)')) \to ((H_v^0)'', \sigma((H_v^0)'', (H_v^0)')$ is continuous, [20, §8.6], that is, $(C_t)''\colon (H_v^\infty, \sigma(H_v^\infty, X)) \to (H_v^\infty, \sigma(H_v^\infty, X))$ is continuous, it follows that $(C_t)'' f_n \to (C_t)'' f$ in $(H_v^\infty, \sigma(H_v^\infty, X))$ as $n \to \infty$. Now, $(f_n)_{n\in\mathbb{N}_0} \subset H^\infty \subseteq H_v^0$, as each f_n is a polynomial, and $(C_t)'' f_n = C_t f_n$ for every $n \in \mathbb{N}_0$. Moreover, the sequence $C_t f_n \to (C_t)'' f$ in $(H(\mathbb{D}), \tau_p)$ as $n \to \infty$. Thus, $(C_t)'' f = C_t f$ as desired. □

Proposition 2.7 *Let v be a weight function satisfying $\lim_{r\to 1^-} v(r) = 0$. For each $t \in [0, 1)$, both of the operators $C_t\colon H_v^\infty \to H_v^\infty$ and $C_t \to H_v^0 \to H_v^0$ are compact.*

Proof Fix $t \in [0, 1)$. Since H_v^0 is a closed subspace of H_v^∞ and $C_t(H_v^0) \subseteq H_v^0$ (cf. Corollary 2.5), it suffices to show that $C_t\colon H_v^\infty \to H_v^\infty$ is compact. First we establish the following Claim:

(*) Let the sequence $(f_n)_{n\in\mathbb{N}} \subset H_v^\infty$ satisfy $\|f_n\|_{\infty,v} \leq 1$ for every $n \in \mathbb{N}$ and $f_n \to 0$ in $(H(\mathbb{D}), \tau_c)$ for $n \to \infty$. Then $C_t f_n \to 0$ in H_v^∞.

To prove the Claim, let $(f_n)_{n\in\mathbb{N}} \subset H_v^\infty$ be a sequence as in (*). Fix $\varepsilon > 0$ and select $\delta \in (0, \beta)$, where $\beta := \min\{1, \frac{\varepsilon(1-t)}{2}, \frac{\varepsilon(1-t)}{2v(0)}\}$. Since $\{\xi \in \mathbb{C}\ |\xi| \leq (1-\delta)\}$ is a compact subset of $\mathbb{D}$, there exists $n_0 \in \mathbb{N}$ such that

$$\max_{|\xi|\leq 1-\delta} |f_n(\xi)| < \delta, \quad n \geq n_0.$$

Recall that $C_t f_n(0) = f_n(0)$ for every $n \in \mathbb{N}$. For $z \in \mathbb{D}\setminus\{0\}$ we have seen previously that

$$v(z)|C_t f_n(z)| = v(z)\left|\int_0^1 \frac{f_n(sz)}{1-stz}ds\right| \le v(z)\int_0^{1-\delta} \frac{|f_n(sz)|}{|1-stz|}ds + v(z)\int_{1-\delta}^1 \frac{|f_n(sz)|}{|1-stz|}ds.$$

Denote the first (resp., second) summand in the right-side of the previous inequality by (A_n) (resp., by (B_n)). Using the facts that $|1-stz| \ge 1-st|z| \ge \max\{1-s, 1-t, 1-|z|\}$, for all $s, t \in [0,1)$ and $z \in \mathbb{D}$, and that v is non-increasing on $[0,1)$ it follows, for every $n \ge n_0$, that $\int_0^{1-\delta} |f_n(sz)|\, ds \le (1-\delta)\max_{|\xi|\le(1-\delta)} |f_n(\xi)|$ (as $|sz| \le (1-\delta)$ for all $s \in [0, 1-\delta]$) and hence, that

$$(A_n) \le \frac{v(0)(1-\delta)}{1-t} \max_{|\xi|\le 1-\delta} |f_n(\xi)| < \frac{\varepsilon}{2}.$$

On the other hand, for every $n \ge n_0$, we have (as $\|f_n\|_{\infty,v} = \sup_{\xi\in\mathbb{D}} v(\xi)|f_n(\xi)| \le 1$) that

$$(B_n) = \int_{1-\delta}^1 \frac{v(z)}{v(sz)} \frac{v(sz)|f_n(sz)|}{|1-stz|}\, ds \le \int_{1-\delta}^1 \frac{\|f_n\|_{\infty,v}}{1-t}\, ds \le \frac{\delta}{1-t} < \frac{\varepsilon}{2}.$$

It follows that $\|C_t f_n\|_{\infty,v} < \varepsilon$ for every $n \ge n_0$. That is, $C_t f_n \to 0$ in H_v^∞ for $n \to \infty$ and so (*) is proved.

The compactness of $C_t \in \mathcal{L}(H_v^\infty)$ can be deduced from (*) as follows. Let $(f_n)_{n\in\mathbb{N}} \subset H_v^\infty$ be any bounded sequence. There is no loss of generality in assuming that $\|f_n\|_{\infty,v} \le 1$ for all $n \in \mathbb{N}$. To establish the compactness of $C_t \in \mathcal{L}(H_v^\infty)$ we need to show that $(C_t f_n)_{n\in\mathbb{N}}$ has a convergent subsequence in H_v^∞.

Since $H_v^\infty \subseteq H(\mathbb{D})$ continuously, the sequence $(f_n)_{n\in\mathbb{N}}$ is also bounded in the Fréchet–Montel space $H(\mathbb{D})$. Hence, there is a subsequence $g_j := f_{n_j}$, for $j \in \mathbb{N}$, of $(f_n)_{n\in\mathbb{N}}$ and $f \in H(\mathbb{D})$ such that $g_j \to f$ in $H(\mathbb{D})$ with respect to τ_c. In particular, $g_j \to f$ pointwise on $\mathbb{D}$. Since $v(z)|g_j(z)| = v(z)|f_{n_j}(z)| \le 1$ for all $z \in \mathbb{D}$ and $j \in \mathbb{N}$, letting $j \to \infty$ it follows that $v(z)|f(z)| \le 1$ for all $z \in \mathbb{D}$, that is, $f \in H_v^\infty$ with $\|f\|_{\infty,v} \le 1$. Let $h_j := \frac{1}{2}(g_j - f)$, for $j \in \mathbb{N}$. Then $\|h_j\|_{\infty,v} \le 1$, for $j \in \mathbb{N}$, and $h_j \to 0$ in $H(\mathbb{D})$ with respect to τ_c. Condition (*) implies that $C_t h_j \to 0$ in H_v^∞ from which it follows that $C_t f_{n_j} = C_t g_j = C_t(g_j - f) + C_t f = 2C_t h_j + C_t f \to C_t f$ in H_v^∞, as desired. □

Proposition 2.8 *Let v be a weight function on $[0,1)$ satisfying $\lim_{r\to 1^-} v(r) = 0$. For each $t \in [0,1)$ the spectra of $C_t \in \mathcal{L}(H_v^\infty)$ and of $C_t \in \mathcal{L}(H_v^0)$ are given by*

$$\sigma_{pt}(C_t; H_v^\infty) = \sigma_{pt}(C_t; H_v^0) = \left\{\frac{1}{m+1} : m \in \mathbb{N}_0\right\}, \tag{2.1}$$

and

$$\sigma(C_t; H_v^\infty) = \sigma(C_t; H_v^0) = \left\{\frac{1}{m+1} : m \in \mathbb{N}_0\right\} \cup \{0\}. \tag{2.2}$$

Proof Let $t \in [0,1)$ be fixed. By [13, Lemma 3.6] we know that the point spectrum of the operator $C_t^\omega \in \mathcal{L}(\omega)$ is given by $\sigma_{pt}(C_t^\omega; \omega) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$ and, for each $m \in \mathbb{N}_0$, that the corresponding eigenspace $\mathrm{Ker}(\frac{1}{m+1}I - C_t^\omega)$ is 1-dimensional and is generated by an eigenvector $x^{[m]} = (x_n^{[m]})_{n\in\mathbb{N}_0} \in \ell^1$. Since $H_v^0 \subseteq H_v^\infty \subseteq H(\mathbb{D})$ with continuous inclusions and $\Phi\colon H(\mathbb{D}) \to \omega$ (cf. Sect. 1) is a continuous embedding, this implies that $\sigma_{pt}(C_t; H_v^0) \subseteq \sigma_{pt}(C_t; H_v^\infty) \subseteq \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$. Indeed, let $f \in H(\mathbb{D})\setminus\{0\}$ and $\lambda \in \mathbb{C}$ satisfy $C_t f = \lambda f$. Then $\lambda f(z) = \sum_{n=0}^\infty \widehat{(\lambda f)}(n) z^n = \sum_{n=0}^\infty \lambda \hat{f}(n) z^n$ and, by (1.6), we have that $(C_t f)(z) = \sum_{n=0}^\infty (C_t^\omega \hat{f})_n z^n$. It follows that $C_t^\omega \hat{f} = \lambda \hat{f}$ in ω with $\hat{f} \neq 0$ and so $\lambda \in \sigma_{pt}(C_t^\omega; \omega) = \{\frac{1}{m+1}\ m \in \mathbb{N}_0\}$.

To conclude the proof, it remains to show that $\{\frac{1}{m+1} : m \in \mathbb{N}_0\} \subseteq \sigma_{pt}(C_t; H_v^0)$. To establish this recall, for each $m \in \mathbb{N}_0$, that the eigenvector $x^{[m]} \in \ell^1$ and hence, the function $g_m(z) := \sum_{n=0}^{\infty}(x^{[m]})_n z^n$ belongs to H_v^0 because $0 \leq v(z)|g_m(z)| \leq v(z)\|x^{[m]}\|_{\ell^1}$ for $z \in \mathbb{D}$ and $\lim_{r\to 1^-} v(r) = 0$. Moreover, according to (1.5) and (1.6) we have, for each $z \in \mathbb{D}$, that

$$C_t g_m(z) = \sum_{n=0}^{\infty}(C_t^{\omega} x^{[m]})_n z^n = \sum_{n=0}^{\infty}(\frac{1}{m+1} x^{[m]})_n z^n = \frac{1}{m+1}\sum_{n=0}^{\infty}(x^{[m]})_n z^n = \frac{1}{m+1} g_m(z).$$

Thus g_m is an eigenvector of $C_t \in \mathcal{L}(H_v^0)$ corresponding to the eigenvalue $\frac{1}{m+1}$.

The validity of $\sigma(C_t; H_v^0) = \sigma(C_t; H_v^{\infty}) = \{\frac{1}{m+1} : m \in \mathbb{N}_0\} \cup \{0\}$ follows from the fact that C_t is a compact operator on both spaces. □

We now investigate the norm of C_t on H_v^{∞} for the standard weights $v_\gamma(z) := (1-|z|)^\gamma$, for $\gamma > 0$ and $z \in \mathbb{D}$, which satisfy $\lim_{r\to 1^-} v_\gamma(r) = 0$.

Proposition 2.9 *Let $t \in (0,1)$ and $\gamma > 0$.*

(i) *The operator norm $\|C_t\|_{H_{v_\gamma}^\infty \to H_{v_\gamma}^\infty} = 1$, for every $\gamma \geq 1$.*
(ii) *For each $\gamma \in (0,1)$, the inequality $\|C_t\|_{H_{v_\gamma}^\infty \to H_{v_\gamma}^\infty} \leq \min\{-\frac{\log(1-t)}{t}, \frac{1}{\gamma}\}$ is valid.*

Proof We adapt the arguments given for the Cesàro operator C_1 in the proof of [2, Theorem 2.3].

Let $\gamma > 0$ and $t \in (0,1)$ be fixed. For $f \in H_{v_\gamma}^\infty$ with $\|f\|_{\infty, v_\gamma} = 1$ we have

$$\begin{aligned} |C_t f(z)| &= \frac{1}{|z|}\left|\int_0^1 \frac{f(sz)}{1-stz}ds\right| \leq \int_0^1 \frac{|f(sz)|}{1-st|z|}ds \\ &\leq \int_0^1 \frac{|f(sz)|}{1-s|z|}ds \leq \int_0^1 \frac{ds}{(1-s|z|)^{\gamma+1}} = \frac{1}{(1-|z|)^\gamma}\frac{1-(1-|z|)^\gamma}{\gamma|z|}, \end{aligned}$$

as $z \in \mathbb{D}$ implies that $1 - st|z| \geq 1 - s|z|$, for $s \in (0,1)$. Accordingly,

$$v_\gamma(z)|C_t f(z)| = (1-|z|)^\gamma |C_t f(z)| \leq \frac{1-(1-|z|)^\gamma}{\gamma|z|}, \quad z \neq 0,$$

and hence,

$$\|C_t f\|_{\infty, v_\gamma} \leq \frac{1}{\gamma}\sup_{z\in\mathbb{D}}\frac{1-(1-|z|)^\gamma}{|z|}.$$

Define $\phi(s) := \frac{1-(1-s)^\gamma}{s}$ for $s \in (0,1]$ and $\phi(0) = \gamma$, in which case ϕ is continuous. So, the previous inequality yields $\|C_t f\|_{\infty, v_\gamma} \leq \frac{M_\gamma}{\gamma}$, for all $\|f\|_{\infty, v_\gamma} \leq 1$, that is, $\|C_t\|_{H_{v_\gamma}^\infty \to H_{v_\gamma}^\infty} \leq \frac{M_\gamma}{\gamma}$, where $M_\gamma := \sup_{s\in[0,1]}\phi(s)$. Proposition 2.4 yields that $1 \leq \|C_t\|_{H_{v_\gamma}^\infty \to H_{v_\gamma}^\infty} \leq -\frac{\log(1-t)}{t}$ for $t \in (0,1)$. On page 101 of [2] it is shown that $\frac{M_\gamma}{\gamma} \leq 1$ whenever $\gamma \geq 1$ and that $M_\gamma \leq 1$ for all $\gamma \in (0,1)$. The proof of both parts (i) and (ii) follows immediately. □

Remark 2.10 For each $\gamma > 0$ let $v_\gamma(z) = (1-|z|)^\gamma$, for $z \in \mathbb{D}$. Proposition 2.9 implies that $\sup_{0\leq t<1}\|C_t\|_{H_{v_\gamma}^\infty \to H_{v_\gamma}^\infty} < \infty$. Moreover, if $\gamma \geq 1$, then $\|C_t^n\|_{H_{v_\gamma}^\infty \to H_{v_\gamma}^\infty} = 1$ for every $n \in \mathbb{N}$; see case (i) in the proof of [2, Theorem 2.3] together with the fact that $1 \in \sigma_{pt}(C_t, H_{v_\gamma}^\infty)$ by Proposition 2.8.

Let $n \in \mathbb{N}$ be fixed. Consider the weight $v(z) = (\log \frac{e}{1-|z|})^{-n}$, for $z \in \mathbb{D}$, which satisfies $v(0) = 1$ and $\lim_{|z|\to 1^-} v(z) = 0$.

The function $f(z) := [\log(1-z)]^n \in H(\mathbb{D})$ belongs to H_v^∞. Indeed, for each $z \in \mathbb{D}$, we have that

$$|\log(1-z)| = \left| -\sum_{n=1}^{\infty} \frac{z^n}{n} \right| \le \sum_{n=1}^{\infty} \frac{|z|^n}{n} = -\log(1-|z|)$$

and hence, that $|f(z)| = |\log(1-z)|^n \le (-\log(1-|z|))^n$. Since v is given by $v(z) = (1-\log(1-|z|))^{-n}$ and $\lim_{|z|\to 1^-} \frac{-\log(1-|z|)}{1-\log(1-|z|)} = 1$, it follows that $\|f\|_{\infty,v} < \infty$ and so $f \in H_v^\infty$. On the other hand,

$$C_1 f(z) = \frac{1}{z}\int_0^z \frac{(\log(1-\xi)^n)}{1-\xi} d\xi = -\frac{1}{(n+1)z}(\log(1-z))^{n+1}, \quad z \in \mathbb{D}.$$

Accordingly, $C_1 f \notin H_v^\infty$ since

$$\begin{aligned} \lim_{s\to s^-} v(s)|(C_1)f(s)| &= \frac{1}{n+1}\lim_{s\to 1^-}\left|\frac{(\log(1-s))^{n+1}}{s(1-\log(1-s))^n}\right| \\ &= \frac{1}{n+1}\lim_{s\to 1^-}\left|\left(\frac{\log(1-s)}{1-\log(1-s)}\right)^n \frac{\log(1-s)}{s}\right| = \infty. \end{aligned}$$

This implies that the Cesàro operator C_1 is not well-defined on H_v^∞, that is, $C_1(H_v^\infty) \not\subseteq H_v^\infty$. But, by Proposition 2.4 the generalized Cesàro operator $C_t \in \mathcal{L}(H_v^\infty)$ for every $t \in [0, 1)$. At this point, the following question arises: *Is* $\sup_{t\in[0,1)} \|C_t\|_{H_v^\infty \to H_v^\infty} < \infty$ *for this particular* v? Our next two results show that the answer is negative for certain weights v, which includes $v(z) = \left(\log \frac{e}{1-|z|}\right)^{-n}$ for $z \in \mathbb{D}$.

Proposition 2.11 *Let* v *be a weight function on* $[0, 1)$ *such that* $\sup_{t\in[0,1)} \|C_t\|_{H_v^\infty\to H_v^\infty} < \infty$. *Then* $C_1 \in \mathcal{L}(H_v^\infty)$.

Proof Proposition 2.1 implies that $\{C_t : t \in [0, 1)\}$ is equicontinuous in $\mathcal{L}(H(\mathbb{D}))$. The claim is that $\lim_{t\to 1^-} C_t f(z) = C_1 f(z)$, for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

To prove this claim fix $f \in H(\mathbb{D})$ and $z \in \mathbb{D}\setminus\{0\}$. Recall, for $t \in [0, 1)$, that

$$C_t f(z) = \frac{1}{z}\int_0^z \frac{f(\xi)}{1-t\xi} d\xi = \int_0^1 \frac{f(sz)}{1-stz} ds$$

and

$$C_1 f(z) = \frac{1}{z}\int_0^z \frac{f(\xi)}{1-\xi} d\xi = \int_0^1 \frac{f(sz)}{1-sz} ds.$$

Moreover, for each $z \in \mathbb{D}\setminus\{0\}$, we have (as $|1-stz| \ge (1-|z|)$) that

$$\left|\frac{f(sz)}{1-stz}\right| \le \frac{|f(sz)|}{1-|z|} \le \frac{1}{1-|z|}\max_{|\xi|\le|z|}|f(\xi)|, \quad s \in [0, 1],$$

and that $\lim_{t\to 1^-} \frac{f(sz)}{1-stz} = \frac{f(sz)}{1-sz}$ for every $s \in [0, 1]$. So, we can apply the dominated convergence theorem to conclude that $\lim_{t\to 1^-} C_t f(z) = C_1 f(z)$ for $z \in \mathbb{D}\setminus\{0\}$. For $z = 0$ we have $C_t f(0) = f(0) = C_1 f(0)$ for each $f \in H(\mathbb{D})$ and $t \in [0, 1)$. So, for each $f \in H(\mathbb{D})$, we can conclude that $C_t f \to C_1 f$ pointwise on $\mathbb{D}$ for $t \to 1^-$. The claim is thereby established.

We now show that $C_t f \to C_1 f$ in $H(\mathbb{D})$ as $t \to 1^-$ for every $f \in H_v^\infty$. The assumption $\sup_{t\in[0,1)} \|C_t\|_{H_v^\infty \to H_v^\infty} < \infty$ implies that there exists $M > 0$ satisfying $\|C_t\|_{H_v^\infty \to H_v^\infty} \le M$ for every $t \in [0, 1)$. Therefore,

$$\sup_{z\in\mathbb{D}} |C_t f(z)|v(z) \le M\|f\|_{\infty,v}, \quad f \in H_v^\infty,\ t \in [0, 1). \tag{2.3}$$

Fix $f \in H_v^\infty$. Then $\{C_t f : t \in [0, 1)\}$ is a bounded set in $H(\mathbb{D})$. Indeed, given $r \in (0, 1)$ and $t \in [0, 1)$ we have (as $v(r) \le v(z)$ for all $|z| \le r$) that

$$q_r(C_t f) = \sup_{|z|\le r} |C_t f(z)| = \max_{|z|=r} |C_t f(z)| \le \frac{M}{v(r)}\|f\|_{\infty,v}.$$

So, the set $\{C_t f : t \in [0, 1)\}$ is bounded in the Fréchet–Montel space $H(\mathbb{D})$ and hence, it is relatively compact in $H(\mathbb{D})$. Since $C_t f \to C_1 f$ pointwise on $\mathbb{D}$ for $t \to 1^-$, it follows that $C_t f \to C_1 f$ with respect to τ_c, that is, in the Fréchet space $H(\mathbb{D})$, for $t \to 1^-$. In particular, $C_1 f \in H(\mathbb{D})$.

Since $H_v^\infty \subseteq H(\mathbb{D})$ and $C_t h \to C_1 h$ pointwise on $\mathbb{D}$ as $t \to 1^-$, for every $h \in H(\mathbb{D})$, letting $t \to 1^-$ in (2.3) it follows that

$$|C_1 f(z)|v(z) \le M\|f\|_{\infty,v}, \quad z \in \mathbb{D},$$

that is, $\|C_1 f\|_{\infty,v} \le M\|f\|_{\infty,v}$. But, $f \in H_v^\infty$ is arbitrary and so $C_1 \in \mathcal{L}(H_v^\infty)$. □

Proposition 2.12 *For each* $n \in \mathbb{N}$, *let* $v(z) = (\log(\frac{e}{1-|z|}))^{-n}$ *for* $z \in \mathbb{D}$. *Then* $\sup_{t\in[0,1)} \|C_t\|_{H_v^\infty \to H_v^\infty} = \infty$.

Proof Apply Proposition 2.11 and the discussion prior it. □

3 Linear dynamics and mean ergodicity of C_t

The aim of this section is to investigate the mean ergodicity and the linear dynamics of the operators C_t, for $t \in [0, 1)$, acting on $H(\mathbb{D})$, H_v^∞ and H_v^0

An operator $T \in \mathcal{L}(X)$, with X a lcHs, is called *power bounded* if $\{T^n : n \in \mathbb{N}_0\}$ is an equicontinuous subset of $\mathcal{L}(X)$. For a Banach space X, this means that $\sup_{n\in\mathbb{N}_0} \|T^n\|_{X\to X} < \infty$. Given $T \in \mathcal{L}(X)$, the averages

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N},$$

are usually called the Cesàro means of T. The operator T is said to be *mean ergodic* (resp., *uniformly mean ergodic*) if $(T_{[n]})_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp., in $\mathcal{L}_b(X)$). It is routine to check that $\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n}T_{[n-1]}$, for $n \ge 2$, and hence, $\tau_s\text{-}\lim_{n\to\infty} \frac{T^n}{n} = 0$ whenever T is mean ergodic. Every power bounded operator on a Fréchet–Montel space X is necessarily uniformly mean ergodic, [1, Proposition 2.8]. Concerning the linear dynamics of $T \in \mathcal{L}(X)$, with X a lcHs, the operator T is called *supercyclic* if, for some $z \in X$, the projective orbit $\{\lambda T^n z : \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X. Since the closure of the linear span of a projective orbit is separable, if such a supercyclic operator $T \in \mathcal{L}(X)$ exists, then X is necessarily separable.

Observe that the space H_v^∞ is never separable, [24, Theorem 1.1]. Therefore, every operator $T \in \mathcal{L}(H_v^\infty)$ is clearly not supercyclic. However, the spaces $H(\mathbb{D})$, [21, Theorem 27.2.5], and H_v^0, [24, Theorem 1.1], for every weight v are always separable. Hence, the problem of supercyclicity for non-zero operators $T \in \mathcal{L}(H(\mathbb{D}))$ and $T \in \mathcal{L}(H_v^0)$ arises.

The following result, [5, Theorem 6.4], is stated here for Banach spaces.

Theorem 3.1 *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be a compact operator such that $1 \in \sigma(T; X)$ with $\sigma(T; X)\backslash\{1\} \subseteq \overline{B(0, \delta)}$ for some $\delta \in (0, 1)$ and satisfying* $\mathrm{Ker}(I - T) \cap \mathrm{Im}(I - T) = \{0\}$. *Then T is power bounded and uniformly mean ergodic.*

A consequence of the previous theorem is the following result.

Proposition 3.2 *Let v be a weight function on* $[0, 1)$ *satisfying* $\lim_{r\to 1^-} v(r) = 0$. *For each $t \in [0, 1)$ both of the operators $C_t \in \mathcal{L}(H_v^\infty)$ and $C_t \in \mathcal{L}(H_v^0)$ are power bounded, uniformly mean ergodic and fail to be supercyclic.*

Proof Fix $t \in [0, 1)$. It was already noted that $C_t \in \mathcal{L}(H_v^\infty)$ cannot be supercyclic. The operator C_t is a compact operator on both H_v^∞ and on H_v^0 (cf. Proposition 2.7). Therefore, the compact transpose operators $C_t' \in \mathcal{L}((H_v^\infty)')$ and $C_t' \in \mathcal{L}((H_v^0)')$ have the same non-zero eigenvalues as C_t (see, e.g., [15, Theorem 9.10-2(2)]). In view of Proposition 2.8 it follows that $\sigma_{pt}(C_t'; (H_v^\infty)') = \sigma_{pt}(C_t'; (H_v^0)') = \{\frac{1}{m+1} : m \in \mathbb{N}_0\}$. We can apply [6, Proposition 1.26] to conclude that C_t is not supercyclic on H_v^0.

By Proposition 2.8 and its proof (as $x^{[0]} = (t^n)_{n\in\mathbb{N}_0}$) we have that $\mathrm{Ker}(I - C_t) = \mathrm{span}\{g_0\}$, with $g_0(z) = \sum_{n=0}^\infty t^n z^n$, for $z \in \mathbb{D}$. On the other hand, $\mathrm{Im}(I - C_t)$ is a closed subspace of H_v^∞ (resp., of H_v^0), as C_t is compact in H_v^∞ (resp., in H_v^0)), and $\mathrm{Im}(I - C_t) \subseteq \{g \in H_v^\infty : g(0) = 0\}$ (resp., $\subseteq \{g \in H_v^0 : g(0) = 0\}$), because $C_t f(0) = f(0)$ for each $f \in H_v^\infty$ (resp., each $f \in H_v^0$). Moreover, [15, Theorem 9.10.1] implies that $\mathrm{codim}\,\mathrm{Im}(I - C_t) = \dim \mathrm{Ker}(I - C_t) = 1$. Accordingly, both $\mathrm{Im}(I - C_t)$ and $\{g \in H_v^\infty : g(0) = 0\} = \mathrm{Ker}(\delta_0)$ are hyperplanes, where $\delta_0 \in (H_v^\infty)'$ is the linear evaluation functional $f \mapsto f(0)$, for $f \in H_v^\infty$. It follows that necessarily $\mathrm{Im}(I - C_t) = \{g \in H_v^\infty : g(0) = 0\}$.

Let $h \in \mathrm{Im}(I - C_t) \cap \mathrm{Ker}(I - C_t)$. Then $h(0) = 0$ and there exists $\lambda \in \mathbb{C}$ such that $h = \lambda g_0$. This yields that $0 = h(0) = \lambda g_0(0) = \lambda$. Hence, $h = 0$. So, $\mathrm{Im}(I - C_t) \cap \mathrm{Ker}(I - C_t) = \{0\}$.

Proposition 2.8 implies that $1 \in \sigma(C_t; H_v^\infty) = \sigma(C_t; H_v^0) = \{\frac{1}{m+1} ; m \in \mathbb{N}_0\} \cup \{0\}$. Consequently, for $\delta = \frac{1}{2}$, all the assumptions of Theorem 3.1 are satisfied. So, we can conclude that C_t is power bounded and uniformly mean ergodic on both H_v^∞ and on H_v^0. □

In contrast to the compactness of C_t acting in the Banach spaces H_v^∞ and H_v^0 (cf. Proposition 2.7) the situation for the Fréchet space $H(\mathbb{D})$ is different.

Proposition 3.3 *For each $t \in [0, 1)$ the operator $C_t : H(\mathbb{D}) \to H(\mathbb{D})$ is an isomorphism and, hence, it is not compact.*

Proof Fix $t \in [0, 1)$. Consider the operator $T_t : H(\mathbb{D}) \to H(\mathbb{D})$, for $f \in H(\mathbb{D})$, given by

$$T_t f(z) := (1 - tz)(zf(z))' = (1 - tz)(f(z) + zf'(z)), \quad z \in \mathbb{D}.$$

Then T_t is clearly well-defined. Moreover, its graph is closed. Indeed, for a given sequence $(f_n)_{n\in\mathbb{N}} \subset H(\mathbb{D})$, suppose that $f_n \to f$ in $H(\mathbb{D})$ and $T_t f_n \to g$ in $H(\mathbb{D})$. Since multiplication operators (by elements from $H(\mathbb{D})$) and the differentiation operator are continuous on $H(\mathbb{D})$ and the evaluation functionals at points of $\mathbb{D}$ belong to $H(\mathbb{D})'$, it follows that $f_n' \to f'$ in $H(\mathbb{D})$ and hence, $T_t f_n = (1-tz)(f_n + zf_n') \to (1-tz)(f + zf') = T_t f$ in $H(\mathbb{D})$. Accordingly, $g = T_t f$. Since $H(\mathbb{D})$ is a Fréchet space, the closed graph theorem, [20, Corollary 5.4.3], implies that $T_t \in \mathcal{L}(H(\mathbb{D}))$.

Finally, it is routine to verify that $C_t \circ T_t = T_t \circ C_t = I$. So, the inverse operator $C_t^{-1} = T_t \in \mathcal{L}(H(\mathbb{D}))$ exists and hence, C_t is a bi-continuous isomorphism of $H(\mathbb{D})$ onto itself. In particular, C_t cannot be compact. □

Let $\Lambda := \{\frac{1}{n+1} : n \in \mathbb{N}_0\}$ and $\Lambda_0 := \Lambda \cup \{0\}$. We recall from [4, Lemma 2.7] the following lemma, which is an extension of a result of Rhoades [27].

Lemma 3.4 *For every $\mu \in \mathbb{C}\backslash\Lambda_0$ there exist $\delta = \delta_\mu > 0$ and constants $d_\delta, D_\delta > 0$ such that $\overline{B(\mu,\delta)} \cap \Lambda_0 = \emptyset$ and*

$$\frac{d_\delta}{n^{\alpha(\nu)}} \le \prod_{k=1}^{n} \left|1 - \frac{1}{k\nu}\right| \le \frac{D_\delta}{n^{\alpha(\nu)}}, \quad \forall n \in \mathbb{N},\ \nu \in B(\mu,\delta), \tag{3.1}$$

where $\alpha(\nu) := \mathrm{Re}(\frac{1}{\nu})$.

Remark 3.5 As a direct application of Lemma 3.4 we obtain, for every $\mu \in \mathbb{C}\backslash\Lambda_0$, that there exist $\delta > 0$ and $d_\delta, D_\delta > 0$ such that $\overline{B(\mu,\delta)} \cap \Lambda_0 = \emptyset$ and, for every $\nu \in B(\mu,\delta)$ and $n \in \mathbb{N}_0$, we have that

$$d_\delta D_\delta^{-1} \left(\frac{n-h}{n+1}\right)^{\alpha(\nu)} \le \prod_{j=n-h+1}^{n+1} \left|1 - \frac{1}{j\nu}\right| \le D_\delta d_\delta^{-1} \left(\frac{n-h}{n+1}\right)^{\alpha(\nu)}, \tag{3.2}$$

for all $h \in \{1, \ldots, n-1\}$, where $\alpha(\nu) = \mathrm{Re}(\frac{1}{\nu})$.

For each $k \in \mathbb{N}$ with $k \ge 2$ define $r_k := (1 - \frac{1}{k})$. Define the norms $\|\cdot\|_k$ and $|||\cdot|||_k$ on $H(\mathbb{D})$ by

$$\|f\|_k := \sum_{n=0}^{\infty} |\hat{f}(n)| r_k^n, \quad f = \sum_{n=0}^{\infty} \hat{f}(n) z^n,$$

and

$$|||f|||_k := \sup_{n \in \mathbb{N}_0} |\hat{f}(n)| r_k^n \quad f = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$

Lemma 3.6 *Each of the sequences $\{\|\cdot\|_k\}_{k\ge2}$ and $\{|||\cdot|||_k\}_{k\ge2}$ is a fundamental system of norms for $(H(\mathbb{D}), \tau_c)$.*

Proof Given $r \in (0,1)$ choose any $k \ge 2$ such that $0 < r < (1 - \frac{1}{k})$. Then, for every $f \in H(\mathbb{D})$, we have

$$q_r(f) = \sup_{|z|=r} \left|\sum_{n=0}^{\infty} \hat{f}(n) z^n\right| \le \sum_{n=0}^{\infty} |\hat{f}(n)| r^n \le \sum_{n=0}^{\infty} |\hat{f}(n)| \left(1 - \frac{1}{k}\right)^n = \|f\|_k.$$

On the other hand, given $k \ge 2$, let $r_k := (1 - \frac{1}{k}) < (1 - \frac{1}{k+1}) := r_{k+1}$. By the Cauchy inequalities, for $n \in \mathbb{N}_0$, we have

$$|\hat{f}(n)| \le \frac{1}{r_{k+1}^n} \max_{|z|=r_{k+1}} |f(z)| = \frac{1}{r_{k+1}^n} q_{r_{k+1}}(f), \quad f \in H(\mathbb{D}),$$

and hence,

$$\|f\|_{r_k} = \sum_{n=0}^{\infty} |\hat{f}(n)| r_k^n \le q_{r_{k+1}}(f) \sum_{n=0}^{\infty} \left(\frac{r_k}{r_{k+1}}\right)^n = c q_{r_{k+1}}(f), \quad f \in H(\mathbb{D}),$$

with $c = \frac{1}{1-\frac{r_k}{r_{k+1}}} = k^2 > 0$ as $\frac{r_k}{r_{k+1}} < 1$, which is independent of f.

So, the systems $\{q_r\}_{r\in(0,1)}$ and $\{\|\cdot\|_k\}_{k\geq 2}$ are equivalent on $H(\mathbb{D})$.

Observe, for every $k \geq 2$, that

$$|||f|||_k = \sup_{n\in\mathbb{N}_0} |\hat{f}(n)|r_k^n \leq \sum_{n=0}^{\infty} |\hat{f}(n)|r_k^n = \|f\|_k, \quad f \in H(\mathbb{D}),$$

and that

$$\|f\|_k = \sum_{n=0}^{\infty} |\hat{f}(n)|r_k^n = \sum_{n=0}^{\infty} |\hat{f}(n)| \left(\frac{r_k}{r_{k+1}}\right)^n r_{k+1}^n$$
$$\leq \sup_{n\in\mathbb{N}_0} |\hat{f}(n)|r_{k+1}^n \sum_{n=0}^{\infty} \left(\frac{r_k}{r_{k+1}}\right)^n = k^2|||f|||_{k+1},$$

for $f \in H(\mathbb{D})$, where $\sum_{n=0}^{\infty} \left(\frac{r_k}{r_{k+1}}\right)^n = k^2$. Therefore, the systems $\{\|\cdot\|_k\}_{k\geq 2}$ and $\{|||\cdot|||_k\}_{k\geq 2}$ are equivalent. □

Proposition 3.7 *For each $t \in [0, 1)$ the spectra of the operator $C_t \in \mathcal{L}(H(\mathbb{D}))$ are given by*

$$\sigma_{pt}(C_t; H(\mathbb{D})) = \sigma(C_t; H(\mathbb{D})) = \Lambda \tag{3.3}$$

and

$$\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0. \tag{3.4}$$

Proof Let $t \in [0, 1)$ be fixed. For any weight function v on $[0, 1)$ satisfying $\lim_{r\to 1^-} v(r) = 0$, we have $H_v^\infty \subseteq H(\mathbb{D})$ continuously and $\Phi\colon H(\mathbb{D}) \to \omega$ is a continuous imbedding. Accordingly, $\sigma_{pt}(C_t; H_v^\infty) \subseteq \sigma_{pt}(C_t; H(\mathbb{D})) \subseteq \Lambda$; see the proof of Proposition 2.8. Since $\sigma_{pt}(C_t; H_v^\infty) = \Lambda$ (cf. Proposition 2.8) and $\sigma_{pt}(C_t^\omega; \omega) = \Lambda$ [5, Theorem 3.7], it follows that $\sigma_{pt}(C_t; H(\mathbb{D})) = \Lambda$. Moreover, in view of Proposition 2.8 above and Theorem 3.7 in [5], the eigenspace corresponding to each eigenvalue $\frac{1}{n+1} \in \Lambda$ is 1-dimensional. By Proposition 3.3, the operator $C_t\colon H(\mathbb{D}) \to H(\mathbb{D})$ is a bi-continuous isomorphism and so $0 \notin \sigma(C_t; H(\mathbb{D}))$.

The claim is that $\mathbb{C}\setminus\Lambda_0 \subseteq \rho(C_t; H(\mathbb{D}))$. To establish this claim, fix $\nu \in \mathbb{C}\setminus\Lambda_0$. Given $g(z) = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{D})$, consider the identity

$$(C_t - \nu I)f(z) = g(z), \quad z \in \mathbb{D}, \tag{3.5}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ is to be determined. It follows from (1.6) that $C_t f(z) = \sum_{n=0}^{\infty}(\frac{t^n a_0+t^{n-1}a_1+\cdots+a_n}{n+1})z^n$ from which the identity $(C_t - \nu I)f(z) = \sum_{n=0}^{\infty}(\frac{t^n a_0+t^{n-1}a_1+\cdots+a_n}{n+1} - \nu a_n)z^n$ is clear. So, (3.5) is satisfied if and only if

$$\sum_{n=0}^{\infty} \left(\frac{t^n a_0 + t^{n-1}a_1 + \cdots + a_n}{n+1} - \nu a_n\right) z^n = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{D},$$

that is, if and only if

$$\frac{t^n a_0 + t^{n-1}a_1 + \cdots + a_n}{n+1} - \nu a_n = c_n, \quad n \in \mathbb{N}_0.$$

In view of this we can argue, as in the proof of [5, Lemma 3.6], to show that if a function $f \in H(\mathbb{D})$ exists which satisfies the identity (3.5), then the Taylor coefficients $(a_n)_{n\in\mathbb{N}_0}$ of

f must verify the following equalities

$$\begin{aligned} a_0 &= \frac{c_0}{1-\nu} \\ a_n &= \frac{c_n}{(\frac{1}{n+1}-\nu)} + \sum_{h=1}^{n}(-1)^h \frac{\nu^{h-1}t^h c_{n-h}}{(n+1)\prod_{j=n-h+1}^{n+1}(\frac{1}{j}-\nu)} \\ &=: A_n + B_n, \quad n \geq 1. \end{aligned} \tag{3.6}$$

Observe, for each $n \geq 1$ and $h \in \{1, \ldots, n\}$, that

$$(-1)^h \prod_{j=n-h+1}^{n+1}\left(\frac{1}{j}-\nu\right) = -\prod_{j=n-h+1}^{n+1}\left(\nu-\frac{1}{j}\right) = -\nu^{h+1}\prod_{j=n-h+1}^{n+1}\left(1-\frac{1}{j\nu}\right)$$

and so

$$B_n = -\sum_{h=1}^{n}\frac{\nu^{h-1}t^h c_{n-h}}{\nu^{h+1}(n+1)\prod_{j=n-h+1}^{n+1}(1-\frac{1}{j\nu})} = -\frac{1}{\nu^2}\sum_{h=1}^{n}\frac{t^h c_{n-h}}{(n+1)\prod_{j=n-h+1}^{n+1}(1-\frac{1}{j\nu})}.$$

Accordingly, to verify the claim we need to prove that the power series $\sum_{n=0}^{\infty} a_n z^n$ is convergent in $\mathbb{D}$, with $(a_n)_{n\in\mathbb{N}_0}$ defined according to (3.6). First, observe that the series $g(z) = \sum_{n=0}^{\infty} c_n z^n$ is convergent in $\mathbb{D}$ and satisfies

$$\limsup_{n\to\infty}\sqrt[n]{|c_n|} = \limsup_{n\to\infty}\sqrt[n]{\frac{|c_n|}{|\frac{1}{n+1}-\nu|}} = \limsup_{n\to\infty}\sqrt[n]{|A_n|}.$$

Therefore, the series $\sum_{n=1}^{\infty} A_n z^n$ has the same radius of convergence as the series $\sum_{n=0}^{\infty} c_n z^n$ and hence, it converges in $H(\mathbb{D})$. Accordingly, $f_1(z) := \sum_{n=1}^{\infty} A_n z^n$, for $z \in \mathbb{D}$, belongs to $H(\mathbb{D})$. On the other hand, the series

$$\begin{aligned} \sum_{n=1}^{\infty} B_n z^n &= -\frac{1}{\nu^2}\sum_{n=1}^{\infty}\sum_{h=1}^{n}\frac{t^h c_{n-h}}{(n+1)\prod_{j=n-h+1}^{n+1}(1-\frac{1}{j\nu})} \\ &= -\frac{1}{\nu^2}\sum_{h=1}^{\infty} t^h z^h \sum_{n=h}^{\infty}\frac{c_{n-h}z^{n-h}}{(n+1)\prod_{j=n-h+1}^{n+1}(1-\frac{1}{j\nu})}, \quad z \in \mathbb{D}. \end{aligned}$$

To establish the convergence of the series $\sum_{n=1}^{\infty} B_n z^n$ in $H(\mathbb{D})$, fix $z \in \mathbb{D}\setminus\{0\}$ and $r \in (|z|, 1)$. Recall, for every $n \in \mathbb{N}_0$, that the Taylor coefficients of g satisfy (as $\frac{1}{r} > 1$)

$$|c_n| = \left|\frac{g^{(n)}(0)}{n!}\right| = \left|\frac{1}{2\pi i}\int_{|\xi|=r}\frac{g(\xi)}{\xi^{n+1}}\,d\xi\right| \leq \frac{1}{r^n}\max_{|\xi|=r}|g(\xi)| \leq \frac{C}{r^{n+1}}$$

where $C := \max_{|\xi|=r}|g(\xi)|$. Therefore, setting $\alpha := \alpha(\nu) = \mathrm{Re}(\frac{1}{\nu})$ and $d := d_\delta$ and $D := D_\delta$ for a suitable $\delta > 0$ (cf. Remark 3.5), we obtain via (3.1) and (3.2) that

$$\begin{aligned} &\sum_{h=1}^{\infty} t^h|z|^h \sum_{n=h}^{\infty}\frac{|c_{n-h}|\,|z|^{n-h}}{(n+1)\prod_{j=n-h+1}^{n+1}|1-\frac{1}{j\nu}|} \\ &\leq C\sum_{h=1}^{\infty} t^h|z|^{h-1}\left(\frac{|z|}{r}d^{-1}(h+1)^{-\alpha-1} + \sum_{n=h+1}^{\infty}\left(\frac{|z|}{r}\right)^{n-h+1} Dd^{-1}\left(\frac{n+1}{n-h}\right)^{\alpha}\right) \\ &= Cd^{-1}\sum_{h=1}^{\infty} t^h(h+1)^{-\alpha-1}|z|^h + CDd^{-1}\sum_{h=1}^{\infty} t^h|z|^{h-1}\sum_{n=h+1}^{\infty}\left(\frac{|z|}{r}\right)^{n-h+1}\left(\frac{n+1}{n-h}\right)^{\alpha} \end{aligned}$$

$$\leq Cd^{-1}\left(\sum_{h=1}^{\infty} t^h (h+1)^{-\alpha-1}|z|^h + D\sum_{h=1}^{\infty} t^h |z|^{h-1}\max\{1,(2+h)^{\alpha}\}\sum_{n=h+1}^{\infty}\left(\frac{|z|}{r}\right)^{n-h+1}\right),$$

which is finite after observing that if $\alpha \leq 0$, then $\left(\frac{n+1}{n-h}\right)^{\alpha} = \left(\frac{n-h}{n+1}\right)^{-\alpha} \leq 1$ for every $h \in \mathbb{N}$ and every $n \geq h+1$, whereas if $\alpha > 0$, then $(\frac{n+1}{n-h})^{\alpha} = (1+\frac{h+1}{n-h})^{\alpha} \leq (2+h)^{\alpha}$. This implies that the series $\sum_{n=1}^{\infty} B^n z^n$ converges in $H(\mathbb{D})$. Accordingly, $f_2(z) := \sum_{n=1}^{\infty} B_n z^n$, for $z \in \mathbb{D}$, belongs to $H(\mathbb{D})$.

Set $f(z) := \frac{c_0}{1-\nu} + f_1(z) + f_2(z)$, for $z \in \mathbb{D}$. Then $f \in H(\mathbb{D})$. Moreover, the arguments above imply that f satisfies (3.5). The identities (3.6) imply that f is the unique solution of (3.5). Accordingly, the inverse operator $(C_t - \nu I)^{-1}\colon H(\mathbb{D}) \to H(\mathbb{D})$ exists. In particular, $(C_t - \nu I)^{-1} \in \mathcal{L}(H(\mathbb{D}))$ as it is the inverse of a continuous linear operator on a Fréchet space.

Since $\nu \in \mathbb{C}\setminus\Lambda_0$ is arbitrary and $0 \in \rho(C_t; H(\mathbb{D}))$, we can conclude that $\sigma(C_t; H(\mathbb{D})) = \Lambda$.

It remains to show that $\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0$. To establish this, fix $\mu \in \mathbb{C}\setminus\Lambda_0$ and observe, by Lemma 3.4, that there exist $\delta > 0$ and constants $d_\delta, D_\delta > 0$ such that $\overline{B(\mu,\delta)} \cap \Lambda_0 = \emptyset$ and the inequalities (3.1) and (3.2) are satisfied. We will show that $B(\mu,\delta) \subset \rho(C_t; H(\mathbb{D}))$ and that the set $\{(C_t - \nu I)^{-1} : \nu \in B(\mu,\delta)\}$ is equicontinuous in $\mathcal{L}(H(\mathbb{D}))$. To see this, first observe that the function $\nu \in \overline{B(\mu,\delta)} \mapsto \mathrm{Re}(\frac{1}{\nu}) \in \mathbb{R}$ is continuous and hence, $\alpha_0 := \max_{\nu\in\overline{B(\mu,\delta)}}\{\mathrm{Re}(\frac{1}{\nu})\}$ exists. For the sake of simplicity of notation set $d := d_\delta$ and $D := D_\delta$.

Let $\nu \in B(\mu, r)$, where $r := \frac{1}{2}d(\Lambda_0, \overline{B(\mu,\delta)}) > 0$ has the property that $|\nu - \frac{1}{j}| > r$ for all $j \in \mathbb{N}$. It was proved above, for any fixed $g(z) = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{D})$, that

$$(C_t - \nu I)^{-1} g(z) = \frac{c_0}{1-\nu} + \sum_{n=1}^{\infty}\left(\frac{c_n}{\frac{1}{n+1}-\nu} - \frac{1}{\nu^2}\sum_{h=1}^{n}\frac{(-1)^h t^h c_{n-h}}{(n+1)\prod_{j=n-h+1}^{n+1}(1-\frac{1}{j\nu})}\right) z^n,$$

for each $z \in \mathbb{D}$. So, for $k \geq 2$ fixed, consider the norm $\|\cdot\|_k$ in $H(\mathbb{D})$. Then we have, via (3.6), that

$$\begin{aligned}
&\|(C_t - \nu I)^{-1} g\|_k \\
&\quad\leq \frac{|c_0|}{|1-\nu|} + \sum_{n=1}^{\infty}\left|\frac{c_n}{\frac{1}{n+1}-\nu} - \frac{1}{\nu^2}\sum_{h=1}^{n}\frac{(-1)^h t^h c_{n-h}}{(n+1)\prod_{j=n-h+1}^{n+1}(1-\frac{1}{j\nu})}\right|\left(1-\frac{1}{k}\right)^n \\
&\quad\leq \left(\frac{1}{r}\sum_{n=0}^{\infty}|c_n|\left(1-\frac{1}{k}\right)^n\right) + \frac{1}{|\nu|^2}\sum_{n=1}^{\infty}\sum_{h=1}^{n}\frac{t^h|c_{n-h}|}{(n+1)\prod_{j=n-h+1}^{n+1}|1-\frac{1}{j\nu}|}\left(1-\frac{1}{k}\right)^n \\
&\quad= \frac{1}{r}\|g\|_k + \frac{1}{|\nu|^2}\sum_{h=1}^{\infty} t^h\left(1-\frac{1}{k}\right)^h \sum_{n=h}^{\infty}\frac{|c_{n-h}|}{(n+1)\prod_{j=n-h+1}^{n+1}|1-\frac{1}{j\nu}|}\left(1-\frac{1}{k}\right)^{n-h}.
\end{aligned}$$

Moreover, (3.1) and (3.2) with $\alpha(\nu) = \mathrm{Re}(\frac{1}{\nu}) \leq \alpha_0$ imply, for each $h \in \mathbb{N}$, that

$$\begin{aligned}
&\sum_{n=h}^{\infty}\frac{|c_{n-h}|}{(n+1)\prod_{j=n-h+1}^{n+1}|1-\frac{1}{j\nu}|}\left(1-\frac{1}{k}\right)^{n-h} = \sum_{l=0}^{\infty}\frac{|c_l|}{(l+h+1)\prod_{j=l+1}^{l+h+1}|1-\frac{1}{j\nu}|}\left(1-\frac{1}{k}\right)^l \\
&\quad= \frac{|c_0|}{(h+1)\prod_{j=1}^{h+1}|1-\frac{1}{j\nu}|} + \sum_{l=1}^{\infty}\frac{|c_l|}{(l+h+1)\prod_{j=l+1}^{l+h+1}|1-\frac{1}{j\nu}|}\left(1-\frac{1}{k}\right)^l \\
&\quad\leq d^{-1}|c_0|(h+1)^{\alpha(\nu)-1} + d^{-1}D\sum_{l=1}^{\infty}\frac{|c_l|}{l+h+1}\left(\frac{l+h+1}{l}\right)^{\alpha(\nu)}\left(1-\frac{1}{k}\right)^l
\end{aligned}$$

$$\leq d^{-1}|c_0|(h+1)^{\alpha_0 - 1} + d^{-1} D \sum_{l=1}^{\infty} \frac{|c_l|}{l+h+1} \left(\frac{l+h+1}{l}\right)^{\alpha_0} \left(1 - \frac{1}{k}\right)^l$$
$$\leq \max\{d^{-1}, d^{-1}D\}(2+h)^{\alpha_0} \sum_{l=0}^{\infty} |c_l| \left(1 - \frac{1}{k}\right)^l = K(2+h)^{\alpha_0} \|g\|_k,$$

with $K := \max\{d^{-1}, d^{-1}D\}$, and hence, since $|\nu| > r$ for all $\nu \in B(\mu, \delta)$, that

$$\frac{1}{|\nu|^2} \sum_{h=1}^{\infty} t^h \left(1 - \frac{1}{k}\right)^h \sum_{n=h}^{\infty} \frac{|c_{n-h}|}{(n+1) \prod_{j=n-h+1}^{n+1} |1 - \frac{1}{j\nu}|} \left(1 - \frac{1}{k}\right)^{n-h}$$
$$\leq \frac{K}{r^2} \|g\|_k \sum_{h=1}^{\infty} t^h \left(1 - \frac{1}{k}\right)^h (2+h)^{\alpha_0} = K' \|g\|_k,$$

with $K' = \frac{K}{r^2} \sum_{h=1}^{\infty} t^h \left(1 - \frac{1}{k}\right)^h (2+h)^{\alpha_0} < \infty$, by the ratio test, for instance.

We have established, for every $\nu \in B(\mu, \delta)$, that

$$\|(C_t - \nu I)^{-1} g\|_k \leq (\frac{1}{r} + K')\|g\|_k.$$

Since $g \in H(\mathbb{D})$ and $k \geq 2$ are arbitrary, this shows that the set $\{(C_t - \nu I)^{-1} : \nu \in B(\mu, \delta)\}$ is equicontinuous. Hence, $\sigma^*(C_t; H(\mathbb{D})) = \Lambda_0$. □

Proposition 3.8 *For each $t \in [0, 1)$ the operator $C_t \colon H(\mathbb{D}) \to H(\mathbb{D})$ is power bounded, uniformly mean ergodic but, it fails to be supercyclic. Moreover, $(I - C_t)(H(\mathbb{D}))$ is the closed subspace of $H(\mathbb{D})$ given by*

$$(I - C_t)(H(\mathbb{D})) = \{g \in H(\mathbb{D}) \ : \ g(0) = 0\} \tag{3.7}$$

and we have the decomposition

$$H(\mathbb{D}) = \mathrm{Ker}(I - C_t) \oplus (I - C_t)(H(\mathbb{D})). \tag{3.8}$$

Proof Fix $t \in [0, 1)$. We first prove that C_t is power bounded. Once this is established, C_t is necessarily uniform mean ergodic because $H(\mathbb{D})$ is a Fréchet- Montel space (see [1, Proposition 2.8]).

Given $k \geq 2$ we have, for every $f \in H(\mathbb{D})$ and with $r_k := (1 - \frac{1}{k})$, that

$$|||C_t f|||_k = \sup_{n \in \mathbb{N}_0} \left| \frac{1}{n+1} \sum_{j=0}^{n} t^{n-j} \hat{f}(j) \right| r_k \leq \sup_{n \in \mathbb{N}_0} \frac{1}{n+1} \sum_{j=0}^{n} |\hat{f}(j)| r_k^n$$
$$\leq \sup_{n \in \mathbb{N}_0} \frac{1}{n+1} \sum_{j=0}^{n} |\hat{f}(j)| r_k^j \leq \sup_{j \in \mathbb{N}_0} |\hat{f}(j)| r_k^j = |||f|||_k,$$

because $r_k^n \leq r_k^j$ for all $j \in \{0, 1, \ldots, n\}$. It follows, for every $n \in \mathbb{N}$, that

$$|||C_t^n f|||_k \leq |||f|||_k, \quad f \in H(\mathbb{D}).$$

Since $k \geq 2$ is arbitrary, the operator $C_t \in \mathcal{L}(H(\mathbb{D}))$ is indeed power bounded.

To establish that $C_t \colon H(\mathbb{D}) \to H(\mathbb{D})$ is not supercyclic, note that the continuous embedding $\Phi \colon H(\mathbb{D}) \to \omega$ has dense range. The operator $C_t^{\omega} \in \mathcal{L}(\omega)$ satisfies $\Phi \circ C_t = C_t^{\omega} \circ \Phi$ as an identity in $\mathcal{L}(H(\mathbb{D}), \omega)$, which implies if $C_t \colon H(\mathbb{D}) \to H(\mathbb{D})$ is supercyclic, then also

$C_t^\omega\colon \omega \to \omega$ must be supercyclic as $\Phi \circ C_t^n = \Phi \circ C_t \circ C_t^{n-1} = C_t^\omega \circ \Phi \circ C_t^{n-1} = \cdots = (C_t^\omega)^n \circ \Phi$, for all $n \in \mathbb{N}$, and $\Phi(H(\mathbb{D}))$ is dense in ω. A contradiction with [5, Theorem 6.1].

To establish (3.7) note that $(I - C_t)(H(\mathbb{D})) \subseteq \{g \in H(\mathbb{D}) : g(0) = 0\}$ because $C_t f(0) = f(0)$ for every $f \in H(\mathbb{D})$. To show the reverse inclusion, let $g \in H(\mathbb{D})$ satisfy $g(0) = 0$. Then $h(z) := zg'(z) + g(z)$, for $z \in \mathbb{D}$, is holomorphic and $h(0) = 0$. Accordingly, also $z \mapsto \frac{h(z)}{z}$, for $z \in \mathbb{D}\backslash\{0\}$, and taking the value $h'(0)$ at $z = 0$ is holomorphic in $\mathbb{D}$. Define $f \in H(\mathbb{D})$ by

$$f(z) := \frac{1}{tz - 1} \int_0^z (1 - t\xi) \frac{h(\xi)}{\xi} \, d\xi, \quad z \in \mathbb{D},$$

and note that $f(0) = 0$. Direct calculation reveals that

$$\frac{f(z)}{1 - tz} - (zf(z))' = h(z) = (zg(z))', \quad z \in \mathbb{D},$$

from which it follows that

$$\int_0^z \frac{f(\xi)}{1 - t\xi} \, d\xi - zf(z) = zg(z), \quad z \in \mathbb{D}.$$

Since $f(0) = 0$, we can conclude that

$$\frac{1}{z} \int_0^z \frac{f(\xi)}{1 - t\xi} \, d\xi - f(z) = g(z), \quad z \in \mathbb{D},$$

that is, $(C_t - I)f = g$ and so $g \in (I - C_t)(H(\mathbb{D}))$. Hence, (3.7) is valid.

To show the validity of (3.8) it suffices to repeat the argument given in the proof of Proposition 3.2. □

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